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THE EFFECT OF SHEAR DEFORMATIONS ON THE  
BENDING OF THICK FLANGED CYLINDRICAL  
SHELLS UNDER INTERNAL PRESSURE

Paul Seide

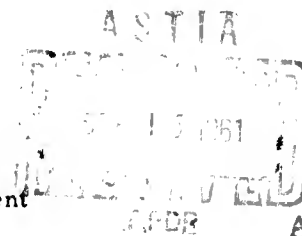
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## I. INTRODUCTION

↓  
A recent problem for which a solution was requested was that of the bending stresses in a thick-walled tube under internal pressure, restrained by heavy integral flanges at its ends and closed by end plates, ~~see Figure 1~~. The arrangement of bolts and the end-plate configuration were such that the flanges could rotate without restraint from the end plates. Although the radius-thickness ratio of the tube was so small that shell theory was almost certainly invalid, it was felt that, in the absence of a more acceptable theory, the inclusion of shear deformation terms in the cylindrical shell and flange equations would yield some useful results. The derivation of the equations used in the investigation is reported ~~herein~~ as well as an illustrative example. It is interesting to note that (for this example) shear deformations were relatively unimportant; the calculated moment increased about eight 8% ~~per cent~~ above that calculated from the equations neglecting shear deformations, due to weakening of the flange relative to the cylinder.

↖

## II. SYMBOLS

a	half-width of flange
D	bending stiffness of cylinder walls $\left[ D = \frac{Et^3}{12(1-\nu^2)} \right]$
$D_R$	bending stiffness of flange considered as a flat plate $\left[ D_R = \frac{E(2a)^3}{12(1-\nu^2)} \right]$
$D_Q$	shear stiffness of cylinder walls $\left[ D_Q = \frac{5}{6} \frac{Et}{2(1+\nu)} \right]$ (reference 2)
$D_{QR}$	shear stiffness of flange considered as a flat plate $\left[ D_{QR} = \frac{5}{6} \frac{E(2a)}{2(1+\nu)} \right]$
d	outer radius of flange
E	Young's modulus of cylinder and flange material
h	radial distance from middle surface of cylinder wall to circle of bolt-hole centers
$K_1$	stiffness defining mean radial displacement through the thickness of flange, due to shear force transferred from cylinder
$K_2$	stiffness defining rotation of inner surface of flange at radius corresponding to cylinder middle surface, due to bending moment transferred from cylinder
$K_2'$	stiffness defining radial displacement of flange inner surface at radius corresponding to cylinder middle surface, due to bending moment transferred from cylinder
$K_3$	stiffness defining rotation of flange inner surface at radius corresponding to cylinder middle surface, due to shear force transferred from cylinder
$K_3'$	stiffness defining radial displacement of flange inner surface at radius corresponding to cylinder middle surface, due to shear force transferred from cylinder
$M_x$	axial moment per unit middle surface perimeter of cylinder
$M_\theta$	circumferential moment per unit length of cylinder
N	axial tensile force per unit middle surface perimeter of cylinder $\left[ N = \frac{pR}{2} \left( 1 - \frac{t}{2R} \right)^2 \right]$
$N_\theta$	circumferential force per unit length of cylinder $\left[ N_\theta = -\frac{Etw_c}{R} \right]$
p	applied internal pressure
$Q_x$	shear force per unit middle surface perimeter of cylinder
R	mean radius of cylinder wall

- $t$  cylinder wall thickness
- $w_c$  inward radial deflection due to bending of cylinder middle surface
- $w_R$  inward average radial deflection of flange at section corresponding to cylinder middle surface, due to shear force transferred from cylinder  $\left(w_R = \frac{Q_x}{K_1}\right)$
- $x$  axial distance along cylinder
- $\delta_c$  outward radial expansion of middle surface of cylinder, due to internal pressure  $p$
- $\delta_R$  outward radial expansion of flange at radius corresponding to middle surface of cylinder, due to pressure  $p$
- $\eta$   $N/D_Q$
- $\phi_R$  rotation of flange inner surface at radius corresponding to middle surface of cylinder, due to bolt moment per unit perimeter,  $Nh$
- $a\phi_R'$  additional outward radial deflection of flange inner surface, due to bending by bolt moment, at radius corresponding to middle surface of cylinder
- $\bar{\phi}_c$  rotation of cylinder wall due to bending  $\left(\bar{\phi}_c = \frac{dw_c}{dx} - \frac{Q_x}{D_Q}\right)$
- $\bar{\phi}_R$  rotation of flange inner surface at radius corresponding to middle surface of cylinder, due to moment and shear force transferred from cylinder  $\left(\bar{\phi}_R = \frac{M_x}{K_2} - \frac{Q_x a}{K_3}\right)$
- $a\bar{\phi}_R'$  additional outward radial deflection of flange inner surface at radius corresponding to cylinder middle surface, due to bending and shear lag by moment and shear force transferred from cylinder  $\left(\bar{\phi}_R' = \frac{M_x}{K_2'} - \frac{Q_x a}{K_3'}\right)$
- $\psi$   $\frac{DEt}{(RD_Q)^2}$

### III. THEORY

The equations for the axisymmetric bending of a cylinder, with shear deformations included, are given by (reference 1)

$$Q_x - \frac{dM_x}{dx} = 0 \quad (1a)$$

$$\frac{dQ_x}{dx} + \frac{N_\theta}{R} + N \frac{d^2 w_c}{dx^2} \quad (1b)$$

$$M_x = -D \left( \frac{d^2 w_c}{dx^2} - \frac{1}{D_Q} \frac{dQ_x}{dx} \right) \quad (1c)$$

$$N_\theta = -\frac{E t w_c}{R} \quad (1d)$$

When we combine Eq. (1), we obtain the following equations for  $w_c$  alone and for  $Q_x$  and  $M_x$  in terms of  $w_c$ :

$$(1 + \eta) \frac{d^4 w_c}{dx^4} - \frac{D_Q}{D} (\psi + \eta) \frac{d^2 w_c}{dx^2} + \left( \frac{D_Q}{D} \right) \psi w_c = 0 \quad (2a)$$

$$Q_x = -D_Q \left[ (1 + \eta) \frac{D}{D_Q} \frac{d^3 w_c}{dx^3} - \psi \frac{dw_c}{dx} \right] \quad (2b)$$

$$M_x = -D_Q \left[ (1 + \eta) \frac{D}{D_Q} \frac{d^2 w_c}{dx^2} - \psi w_c \right] \quad (2c)$$

Equation (2a) may be solved to yield

$$w_c = A_1 \exp\left(\sqrt{\frac{D_Q}{D}} \lambda_1 x\right) + A_2 \exp\left(-\sqrt{\frac{D_Q}{D}} \lambda_1 x\right) + A_3 \exp\left(\sqrt{\frac{D_Q}{D}} \lambda_2 x\right) + A_4 \exp\left(-\sqrt{\frac{D_Q}{D}} \lambda_2 x\right) \quad (3a)$$

where

$$\lambda_{1,2} = \sqrt{\frac{\psi + \eta \pm \sqrt{(\psi - \eta)^2 - 4\psi}}{2(1 + \eta)}} \quad (3b)$$

from which we obtain

$$Q_x = -D_Q \sqrt{\frac{D_Q}{D}} \left\{ \lambda_1 [(1 + \eta) \lambda_1^2 - \psi] \left[ A_1 \exp \left( \sqrt{\frac{D_Q}{D}} \lambda_1 x \right) - A_2 \exp \left( -\sqrt{\frac{D_Q}{D}} \lambda_1 x \right) \right] \right. \\ \left. + \lambda_2 [(1 + \eta) \lambda_2^2 - \psi] \left[ A_3 \exp \left( \sqrt{\frac{D_Q}{D}} \lambda_2 x \right) - A_4 \exp \left( -\sqrt{\frac{D_Q}{D}} \lambda_2 x \right) \right] \right\} \quad (4a)$$

$$M_x = -D_Q \left\{ [(1 + \eta) \lambda_1^2 - \psi] \left[ A_1 \exp \left( \sqrt{\frac{D_Q}{D}} \lambda_1 x \right) + A_2 \exp \left( -\sqrt{\frac{D_Q}{D}} \lambda_1 x \right) \right] \right. \\ \left. + [(1 + \eta) \lambda_2^2 - \psi] \left[ A_3 \exp \left( \sqrt{\frac{D_Q}{D}} \lambda_2 x \right) + A_4 \exp \left( -\sqrt{\frac{D_Q}{D}} \lambda_2 x \right) \right] \right\} \quad (4b)$$

We shall now assume that the cylinder is so long that what happens at one end will not affect the other. We then put

$$A_1 = A_3 = 0 \quad (5)$$

The remaining constants  $A_2$  and  $A_4$  are determined by the conditions obtained by matching cylinder and flange deflections and rotations at a point corresponding to the cylinder middle surface (see Figure 2)

$$\left. \begin{aligned} \delta_c - w_c = \delta_R - w_R + (\bar{\phi}_R' + \bar{\phi}_R') a_3 - \bar{\phi}_c = \bar{\phi}_R + \bar{\phi}_R \end{aligned} \right\} \quad \text{at } x = 0 \quad (6a)$$

or

$$\left. \begin{aligned} w_c - Q_x \left( \frac{1}{K_1} + \frac{a^2}{K_3} \right) + \frac{M_x a}{K_2} = \delta_c - \delta_R - \phi_R' a \\ \frac{dw_c}{dx} - Q_x \left( \frac{1}{D_Q} + \frac{a}{K_3} \right) + \frac{M_x}{K_2} = -\phi_R \end{aligned} \right\} \quad \text{at } x = 0 \quad (6b)$$

Substitution of Eqs. (3), (4), and (5) into Eq. (6b) then yields

$$\left\{ 1 + \left[ \frac{D_Q^3}{D} \left( \frac{1}{K_1} + \frac{a^2}{K_3} \right)^2 \lambda_1 + \frac{aD_Q}{K_2} \right] \left[ \psi - (1+\eta) \lambda_1^2 \right] \right\} A_2 \\ + \left\{ 1 + \left[ \frac{D_Q^3}{D} \left( \frac{1}{K_1} + \frac{a^2}{K_3} \right)^2 \lambda_2 + \frac{aD_Q}{K_2} \right] \left[ \psi - (1+\eta) \lambda_2^2 \right] \right\} A_4 \\ = \delta_c - \delta_R - \phi_R' \quad (7a)$$

$$\left\{ \lambda_1 - \left[ \left( 1 + \frac{aD_Q}{K_3} \right) \lambda_1 + \sqrt{\frac{DD_Q}{K_2^2}} \right] \left[ \psi - (1+\eta) \lambda_1^2 \right] \right\} A_2 \\ + \left\{ \lambda_2 - \left[ \left( 1 + \frac{aD_Q}{K_3} \right) \lambda_2 + \sqrt{\frac{DD_Q}{K_2^2}} \right] \left[ \psi - (1+\eta) \lambda_2^2 \right] \right\} A_4 = \sqrt{\frac{D}{D_Q}} \phi_R \quad (7b)$$

which may be solved to give  $A_2$  and  $A_4$  as

$$A_2 = \frac{1}{(\lambda_2 - \lambda_1) \Delta} \left( \left\{ \lambda_2 - \left[ \left( 1 + \frac{aD_Q}{K_3} \right) \lambda_2 + \sqrt{\frac{DD_Q}{K_2^2}} \right] \left[ \psi - (1+\eta) \lambda_2^2 \right] \right\} (\delta_c - \delta_R - \phi_R') \right. \\ \left. - \left\{ 1 + \left[ \frac{D_Q^3}{D} \left( \frac{1}{K_1} + \frac{a^2}{K_3} \right)^2 \lambda_2 + \frac{aD_Q}{K_2} \right] \left[ \psi - (1+\eta) \lambda_2^2 \right] \right\} \sqrt{\frac{D}{D_Q}} \phi_R \right) \quad (8a)$$

$$A_4 = \frac{-1}{(\lambda_2 - \lambda_1) \Delta} \left( \left\{ \lambda_1 - \left[ \left( 1 + \frac{aD_Q}{K_3} \right) \lambda_1 + \sqrt{\frac{DD_Q}{K_2^2}} \right] \left[ \psi - (1+\eta) \lambda_1^2 \right] \right\} (\delta_c - \delta_R - \phi_R') \right. \\ \left. - \left\{ 1 + \left[ \frac{D_Q^3}{D} \left( \frac{1}{K_1} + \frac{a^2}{K_3} \right)^2 \lambda_1 + \frac{aD_Q}{K_2} \right] \left[ \psi - (1+\eta) \lambda_1^2 \right] \right\} \sqrt{\frac{D}{D_Q}} \phi_R \right) \quad (8b)$$

where

$$\begin{aligned} \Delta = 1 + \left(1 + \frac{aD_Q}{K_3}\right) \left[\eta + \sqrt{\psi(1+\eta)}\right] + \psi \sqrt{\frac{DD_Q}{K_2^2}} \sqrt{\frac{D_Q^3}{D} \left(\frac{1}{K_1} + \frac{a^2}{K_3}\right)^2} \\ + (1+\eta) \left[ \sqrt{\frac{DD_Q}{K_2^2}} + \sqrt{\frac{\psi}{1+\eta}} \sqrt{\frac{D_Q^3}{D} \left(\frac{1}{K_1} + \frac{a^2}{K_3}\right)^2} \right] (\lambda_1 + \lambda_2) \\ + \frac{aD_Q}{K_2} \left[ \sqrt{\psi(1+\eta)} - \psi \frac{aD_Q}{K_3} \right] \end{aligned} \quad (8c)$$

We may now find the end moment from Eq. (4b) as

$$\begin{aligned} M_x|_{x=0} = D_Q \left\{ \left[ \psi - (1+\eta)\lambda_1^2 \right] A_2 + \left[ \psi - (1+\eta)\lambda_2^2 \right] A_4 \right\} \\ = \frac{D_Q}{\Delta} \left\{ \left[ \sqrt{\psi(1+\eta)} - \psi \frac{aD_Q}{K_3} \right] (\delta_c - \delta_R - R'a) - \left[ (1+\eta)(\lambda_1 + \lambda_2) \right. \right. \\ \left. \left. + \psi \sqrt{\frac{D_Q^3}{D} \left(\frac{1}{K_1} + \frac{a^2}{K_3}\right)^2} \right] \sqrt{\frac{D}{D_Q}} \phi_R \right\}. \end{aligned} \quad (9a)$$

The end shear is given similarly by

$$\begin{aligned} Q_x|_{x=0} = \frac{1}{\Delta} \left\{ - \left[ \psi \sqrt{\frac{DD_Q}{K_2^2}} + \sqrt{\frac{\psi}{1+\eta}} (\lambda_1 + \lambda_2) \right] (\delta_c - \delta_R - R'a) \right. \\ \left. + \left[ \eta + \sqrt{\psi(1+\eta)} - \psi \frac{aD_Q}{K_2} \right] \sqrt{\frac{D}{D_Q}} \phi_R \right\} \sqrt{\frac{D_Q^3}{D}} \end{aligned} \quad (9b)$$

It should be noted that the form of  $\lambda_1 + \lambda_2$  depends on whether or not

$$(\psi - \eta)^2 - 4\psi > 0;$$

if it is not  $\lambda_1$  and  $\lambda_2$  are conjugate complex numbers. If

$$(\psi - \eta)^2 - 4\psi > 0$$

$$\lambda_1 + \lambda_2 = \sqrt{\frac{\psi + \eta + \sqrt{(\psi - \eta)^2 - 4\psi}}{2(1 + \eta)}} + \sqrt{\frac{\psi + \eta - \sqrt{(\psi - \eta)^2 - 4\psi}}{2(1 + \eta)}}; \quad (10a)$$

$$\text{but if } (\psi - \eta)^2 - 4\psi < 0$$

$$\lambda_1 + \lambda_2 = \sqrt{2 \sqrt{\frac{\psi}{1 + \eta}} + \frac{\psi + \eta}{1 + \eta}} \quad (10b)$$

When  $D_Q$  becomes infinite we have

$$\Delta = 1 + \frac{a}{K_3} \left[ N + \sqrt{\frac{EtD}{R^2}} \right] + \frac{EtD}{R^2 K_2} \left( \frac{1}{K_1} + \frac{a^2}{K_3'} \right) + \frac{1}{\sqrt{D}} \left[ N + 2 \sqrt{\frac{EtD}{R^2}} \left[ \frac{D}{K_2} + \sqrt{\frac{EtD}{R^2}} \left( \frac{1}{K_1} + \frac{a^2}{K_3'} \right) \right] + \frac{a}{K_2'} \left[ \sqrt{\frac{EtD}{R^2}} - \frac{EtDa}{R^2 K_3'} \right] \right] \quad (11a)$$

$$M_x|_{x=0} = \frac{1}{\Delta} \left\{ \left[ \sqrt{\frac{EtD}{R^2}} - \frac{EtDa}{R^2 K_3'} \right] (\delta_c - \delta_R - \phi_R' a) - \left[ N + 2 \sqrt{\frac{EtD}{R^2}} + \frac{EtD}{R^2} \left( \frac{1}{K_1} + \frac{a^2}{K_3'} \right) \frac{1}{\sqrt{D}} \right] \sqrt{D} \phi_R \right\} \quad (11b)$$

$$Q_y|_{x=0} = \frac{1}{\Delta} \left\{ - \left[ \frac{EtD}{R^2 K_2} + \sqrt{\frac{EtD}{R^2}} \sqrt{\frac{N + 2 \sqrt{\frac{EtD}{R^2}}}{D}} \right] (\delta_c - \delta_R - \phi_R' a) + \left[ N + \sqrt{\frac{EtD}{R^2}} - \frac{EtDa}{R^2 K_2'} \right] \phi_R \right\} \quad (11c)$$

with  $N < 2 \sqrt{\frac{EtD}{R^2}}$  . If  $N > 2 \sqrt{\frac{EtD}{R^2}}$  we replace the quantity

$$\sqrt{N + 2 \sqrt{\frac{EtD}{R^2}}} \text{ by } \left\{ \sqrt{\frac{1}{2} \left[ N + \sqrt{N^2 - \frac{4 EtD}{R^2}} \right]} + \sqrt{\frac{1}{2} \left[ N - \sqrt{N^2 - \frac{4 EtD}{R^2}} \right]} \right\}$$

It will be noted that to determine the junction bending moment and shear, we need values of the initial deflections and rotations  $\delta_c$ ,  $\delta_R$ ,  $\phi_R$ ,  $\phi_R'$  and stiffnesses  $K_1$ ,  $K_2$ ,  $K_3$ ,  $K_2'$ ,  $K_3'$ . The deflections  $\delta_c$  and  $\delta_R$  are readily obtained by elasticity theory (reference 3) as

$$\delta_c = \frac{1+\nu}{E} pR \left(1 - \frac{t}{2R}\right)^2 \left[ (1-\nu) \frac{R}{t} + \frac{1}{2} + \frac{t}{8R} \right] \quad (12a)$$

$$\delta_R = \frac{pR}{E \left[ \left( \frac{d}{R-t} \right)^2 - 1 \right]} \left[ 1 - \nu + (1+\nu) \frac{d^2}{R^2} \right] \quad (12b)$$

The rotations  $\phi_R$  and  $\phi_R'$  are derived in Appendix A to be approximately

$$\begin{aligned} \phi_R = \frac{NR^2}{2D_R} \frac{1}{\left(\frac{d}{R}\right)^2 - 1} & \left[ \frac{1}{1+\nu} \frac{h}{R} \left(2 + \frac{h}{R}\right) + \frac{2}{1-\nu} \left(\frac{d}{R}\right)^2 \ln\left(1 + \frac{h}{R}\right) \right. \\ & \left. + \frac{2D_R}{R^2 D_{QR}} \frac{\left(\frac{d}{R}\right)^2 \frac{h}{R} \left(2 + \frac{h}{R}\right)}{\left(1 + \frac{h}{R}\right)^2} \right] \quad (13a) \end{aligned}$$

$$\phi_R' = \phi_R - \frac{N}{D_{QR}} \quad (13b)$$

The stiffness  $K_1$  is ascertained approximately by assuming the shear force  $Q_x$  to be equivalent to a pressure uniformly distributed over the flange thickness at a section corresponding to the middle surface of the cylinder and the flange to extend only to this section. Then, from reference 3 we have

$$K_1 = \frac{2aE}{R} \frac{\left(\frac{d}{R}\right)^2 - 1}{(1+\nu) \left(\frac{d}{R}\right)^2 + 1 - \nu} \quad (14)$$

If we assume the moment and shear force transmitted to the flange to be equivalent to a single moment applied at the section corresponding to the

middle surface of the cylinder and the flange ~~to~~ extend only to this surface, then elementary plate theory yields

$$\bar{K}_2 = \bar{K}_2' = \bar{K}_3 = \bar{K}_3' = \frac{D_R \left[ \left( \frac{d}{R} \right)^2 - 1 \right]}{R \left[ \frac{1}{1-\nu} \left( \frac{d}{R} \right)^2 + \frac{1}{1+\nu} \right]} \quad (15)$$

which is identical with that obtained from reference 4. If test data or more accurate theoretical values of the deflections, rotations, and stiffnesses are known, these may, of course, be used in place of the approximations given above.

#### IV. ILLUSTRATIVE EXAMPLE

Let us consider a flanged cylinder with the following dimensions and other pertinent properties:

$$\begin{aligned} R &= 2.094 \text{ in.} \\ t &= 1.188 \text{ in.} \\ h &= 2.106 \text{ in.} \\ d &= 5.560 \text{ in.} \\ 2a &= 2.250 \text{ in.} \\ E &= 30 \times 10^6 \text{ psi} \\ \nu &= 0.3 \\ f &= 60,000 \text{ psi.} \end{aligned}$$

We have

$$\begin{aligned} D &= 4.61 \times 10^6 \text{ lb-in.} \\ D_Q &= 11.42 \times 10^6 \text{ lb/in.} \\ N &= 32,226 \text{ lb/in.} \\ \psi &= 0.287 \\ \eta &= 0.00282 \\ (\psi - \eta)^2 - 4\psi &= -1.067 < 0 \\ \lambda_1 + \lambda_2 &= 1.166 \\ \delta_c &= 5.042 \times 10^{-3} \text{ in.} \\ \delta_R &= 3.243 \times 10^{-3} \text{ in.} \\ \phi_R &= 7.404 \times 10^{-3} \\ \phi_R' &= 5.914 \times 10^{-3} \\ K_1 &= 19.77 \times 10^6 \text{ psi} \\ K_2 = K_3 = K_2' = K_3' &= 8.34 \times 10^6 \text{ lb} \end{aligned}$$

and with these quantities, from Eqs. (8c) and (9)

$$M_x \Big|_{x=0} = -18.64 \times 10^3 \text{ lb-in./in.}$$

It is interesting to note that if shear deformations had been neglected, then

$$\phi_R = \phi_R = 6100 \times 10^{-3}$$

we have from Eq. (11),

$$M_x \Big|_{x=0} = -17.17 \times 10^3 \text{ lb-in./in.}$$

which is less than the moment obtained by including shear deformations. If shear deformations in the flange had been neglected, we would have

$$M_x \Big|_{x=0} = -15.53 \times 10^3 \text{ lb-in./in.}$$

We can then conclude that the increase in moment predicted by the inclusion of shear deformations is due to a weakening of the flange with respect to the cylinder, with the result that a larger proportion of the bolt moment is transmitted to the cylinder.

For a cylinder rigidly supported against rotation and deflection, that is

$$\delta_R = \phi_R = \alpha \phi_R' = 0$$

$$K_1 = K_2 = K_3 = K_2' = K_3' = \infty$$

we have

$$M_x \Big|_{x=0} = \frac{D_Q \delta_c}{1 + \sqrt{1 + \eta}}$$

$= 20.06 \times 10^3 \text{ lb-in./in. with shear deformations}$   
 $= 30.85 \times 10^3 \text{ lb-in./in. with } D_Q = \infty$

thus indicating that for some cases the effect of shear deformations can be significant.

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# APPENDIX A

## DETERMINATION OF THE FLANGE ROTATION $\phi_R$ AND DEFLECTION $w_R$

As an approximation, we shall consider the flange to extend only to the middle surface of the cylinder and all reactions to be applied there. It seems unreasonable, in view of all of the approximations of the present analysis, to attempt to consider a more complex loading arrangement. For the determination of the rotation  $\phi_R$  and the deflection  $w_R$ , we thus have a circular plate of outer radius  $d$  loaded by a uniform line load along the radius  $r = R + h$  and reacted at the inner radius  $R$ . The following equations, neglecting the effect of the internal pressure  $p$  and shear force  $Q_x|_{x=0}$  on bending, must then be satisfied (references 1, 5):

$$\frac{d}{dr} (r M_{r1}) - M_{\theta1} - r Q_{r1} = 0 \quad (A1a)$$

$$M_{r1} = -D_R \left[ \frac{d}{dr} \left( \phi_1 - \frac{Q_{r1}}{D_{QR}} \right) + \frac{\nu}{r} \left( \phi_1 - \frac{Q_{r1}}{D_{QR}} \right) \right] \quad (A1b)$$

$$M_{\theta1} = -D_R \left[ \phi_1 - \frac{Q_{r1}}{D_{QR}} + \nu \frac{d}{dr} \left( \phi_1 - \frac{Q_{r1}}{D_{QR}} \right) \right] \quad (A1c)$$

where

$$i = 1, \quad R + \frac{t}{2} \leq r \leq R + h;$$

$$i = 2, \quad R + h \leq r \leq d;$$

$$\phi_i = \frac{dw_i}{dr}$$

and

$$Q_{r1} = -N \frac{R}{r} \quad (A2a)$$

$$Q_{r2} = 0 \quad (A2b)$$

Equations(A1) may be combined to yield

$$\frac{d}{dr} \left\{ \frac{1}{r} \frac{d}{dr} \left[ r \left( \phi_1 - \frac{Q_{r_i}}{D_{Q_R}} \right) \right] \right\} = - \frac{Q_{r_i}}{D_R} \quad (A3)$$

and solved to obtain

$$\phi_1 = A_1 \frac{r}{R} + B_1 \frac{R}{r} + \frac{NR^2}{2D_R} \frac{r}{R} \ln \left( \frac{r}{R} \right) \quad (A4a)$$

$$\phi_2 = A_2 \frac{r}{R} + B_2 \frac{R}{r} \quad (A4b)$$

The boundary conditions to be satisfied are

$$\left. \begin{aligned} M_{r_1} &= 0 & \text{at } r &= R \\ \phi_1 &= \phi_2, \quad M_{r_1} = M_{r_2} & \text{at } r &= R+h \\ M_{r_2} &= 0 & \text{at } r &= d \end{aligned} \right\} \quad (A5)$$

Then

$$(1+\nu) A_1 - (1-\nu) B_1 + \frac{NR^2}{2D_R} - (1-\nu) \frac{N}{D_Q} = 0 \quad (A6a)$$

$$A_1 - A_2 + \frac{B_1 - B_2}{\left(1 + \frac{h}{R}\right)^2} + \frac{NR^2}{2D_R} \ln \left(1 + \frac{h}{R}\right) = 0 \quad (A6b)$$

$$\begin{aligned} (1+\nu)(A_1 - A_2) - \frac{(1-\nu)(B_1 - B_2)}{\left(1 + \frac{h}{R}\right)^2} + \frac{NR^2}{2D_R} \left[ 1 + (1+\nu) \ln \left(1 + \frac{h}{R}\right) \right] \\ - (1-\nu) \frac{N}{D_{Q_R}} - \frac{1}{\left(1 + \frac{h}{R}\right)^2} = 0 \end{aligned} \quad (A6c)$$

$$(1+\nu) A_2 - (1-\nu) \frac{B_2}{\left(\frac{d}{R}\right)^2} \quad (A6d)$$

which may be solved for  $A_1$  and  $B_1$  to yield

$$B_1 = \frac{NR^2}{4D_R} \frac{1}{\frac{d}{R}^2 - 1} \left\{ \left( \frac{d}{R} \right)^2 - \left( 1 + \frac{h}{R} \right)^2 - 2 \frac{1+\nu}{1-\nu} \left( \frac{d}{R} \right)^2 \ln \left( 1 + \frac{h}{R} \right) \right. \\ \left. - \frac{4D}{R^2 D_Q} \left( \frac{d}{R} \right)^2 \left[ 1 - \frac{1+\nu}{2 \left( 1 + \frac{h}{R} \right)^2} - \frac{1-\nu}{2 \left( \frac{d}{R} \right)^2} \right] \right\} \quad (A7a)$$

$$A_1 = \frac{1}{1+\nu} \left[ (1-\nu) \left( B_1 + \frac{N}{D_{QR}} \right) - \frac{NR^2}{2D_R} \right] \quad (A7b)$$

We now have

$$\phi_R - \phi_1|_{r=R} = - (A_1 + B_1) = - \frac{2}{1+\nu} \left[ B_1 + (1-\nu) \frac{N}{2D_{QR}} - \frac{NR^2}{4D_R} \right] \\ = \frac{NR^2}{2D_R} \frac{1}{\frac{d}{R}^2 - 1} \left[ \frac{1}{1+\nu} \frac{h}{R} \left( 2 + \frac{h}{R} \right) + \frac{2}{1-\nu} \left( \frac{d}{R} \right)^2 \ln \left( 1 + \frac{h}{R} \right) \right. \\ \left. + \frac{2D_Q}{R^2 D_{QR}} \frac{\left( \frac{d}{R} \right)^2 \frac{h}{R} \left( 2 + \frac{h}{R} \right)}{1 + \left( \frac{h}{R} \right)^2} \right] \quad (A8)$$

The first part of Eq. (A8) is identical with that given in reference 4, except that Poisson's ratio  $\nu$  equal to  $1/4$  has been used in reference 4. The second part of Eq. (A8), the terms multiplied by  $\frac{1}{D_Q}$ , is the rotation due to

shear deformations. The deflection  $\phi_R' a$  is given, within the framework of plate theory in which straight lines normal to the middle surface of the undeformed plate remain straight, by

$$\phi_R' a = \left( \phi_R - \frac{N}{D_{QR}} \right) a \quad (A9)$$

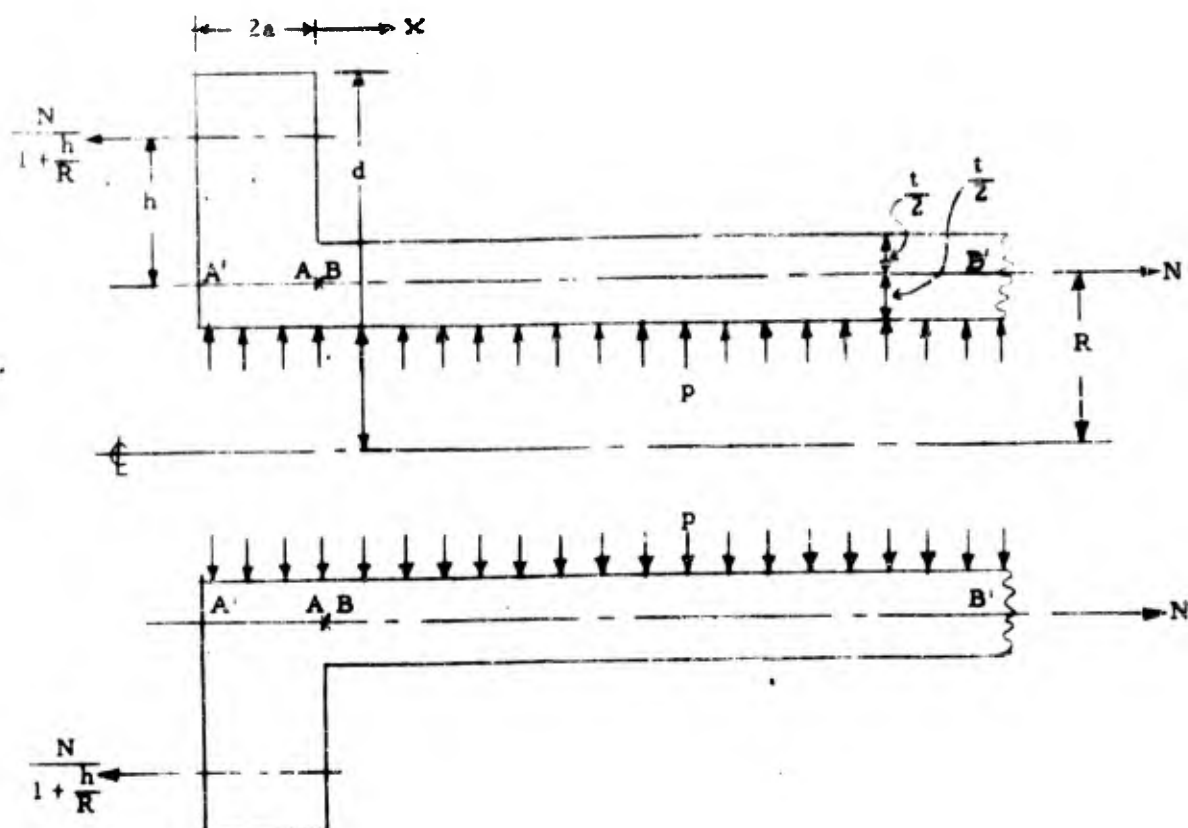
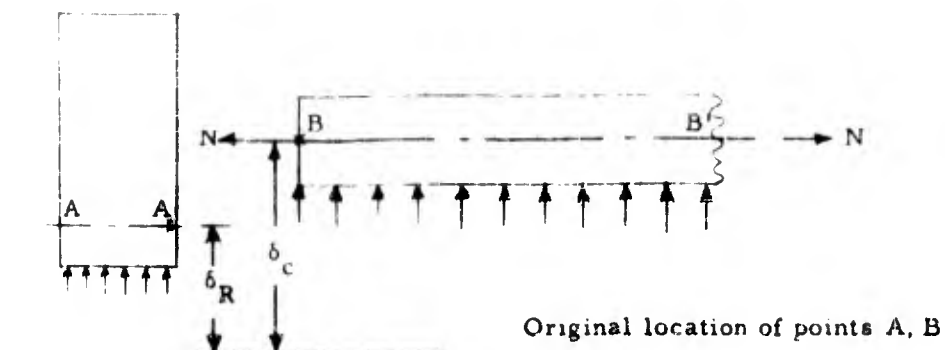
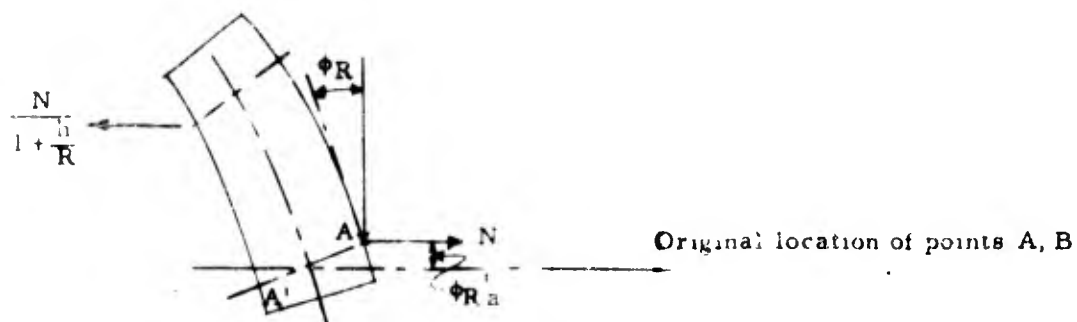


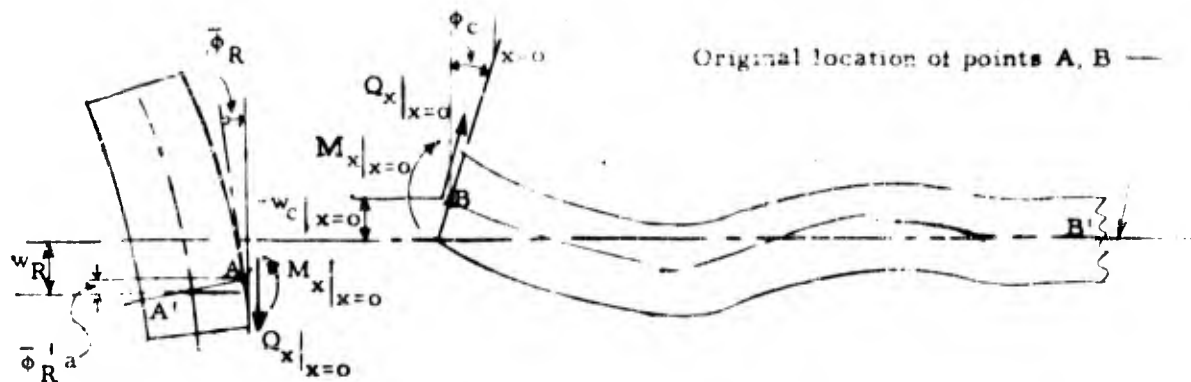
Figure 1. Cross Section of Flanged Pipe Showing Loads and Pertinent Dimensions



(a) Deformations due to internal pressure



(b) Deformations due to bolt moment



(c) Deformations due to bending moment and shear force

Figure 2. Deformations of Cylinder and Flange

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